Proof.—Since M is an irreducible continuum about itself, M contains at least one non-cut point of itself, by theorem 1. Since no continuum is an irreducible continuum about a set consisting of a single point, it follows from theorem 2 that M contains at least two non-cut points.

COROLLARY 2*b*. *Every bounded continuum is an irreducible continuum about the set consisting of all its non-cut points.*[6]

THEOREM 3. *If a continuum (bounded or unbounded) is an irreducible continuum about a set A, then A' contains all points of type 2 of the continuum.*[7]

This theorem is a direct consequence of theorem 2 of the preceding paper. From this theorem and theorems 2 and 4 of the preceding paper follows:

THEOREM 4. *If M is a continuous curve (bounded or unbounded) in space of any finite number of dimensions, which is an irreducible continuum about a set A, then A' contains all the non-cut points of M.*[3]

[1] Presented to the American Mathematical Society at Nashville, Dec. 28, 1927.

[2] H. M. Gehman, these PROCEEDINGS, **12**, 1926 (544–547).

[3] For the case where M is a (bounded) plane continuous curve, see: H. M. Gehman, *Amer. J. Math.*, **49**, 1927 (189–196), theorem 3.

[4] The author wishes to thank Professor W. A. Wilson for suggesting certain simplifications in the proof of theorem 1.

[5] R. L. Moore, *Bull. Amer. Math. Soc.*, **29**, 1923 (289–302), theorem C.

[6] Gehman, these PROCEEDINGS, loc. cit., theorem 4.

[7] See preceding paper for definition of a point of type 2.

CRITERIA FOR THE SIMPLIFICATION OF ALGEBRAIC PLANE CURVES

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It is the purpose of the present note to establish necessary and sufficient conditions for the possibility of effecting certain notable simplifications of a given algebraic plane curve by means of a Cremona transformation. It will be recalled that every such transformation can be factored into the product of a finite number of quadratic transformations and colineations.

Given a curve of order n, which we may suppose irreducible, let it have singular points $P_1P_2 \ldots$ whose actual multiplicity is $r_1r_2 \ldots$, it being understood that these points may be either distinct or infinitely near in the sense commonly used. The theorems used and established are to be birationally invariant, so it is immaterial how near the singular points lie to one another. A curve of order $n-3$ which has a singular point of order $r_i - 1$ or more wherever the original curve has multiplicity r_i is called a *special adjoint of index* 1. If it be of order $n-3k$ with multiplicity at least $r_i - k$ where the original curve has multiplicity r_i, or multiplicity at least 0 when this number is negative, it is called a *special adjoint of index k*. If the system of all such curves be infinite, there may be variable curves, and fixed curves, the general curve being composed of all the fixed curves, perhaps counted several times each, and one variable curve. Such fixed curves are said to be *fundamental,* they have no non-singular intersections with the given curve.

The following theorem, which is now classic,[1] is the basis of all that follows: *If an algebraic plane curve be subjected to a Cremona transformation, the system of special adjoints of each index either goes over into the corresponding system or a part thereof, the rest being a fundamental curve or fundamental curves corresponding to the singular points of the transformation.*

When we are dealing with a quadratic transformation we may introduce or abolish one or more of the sides of the fundamental triangle counted a number of times. The number of new lines introduced is the number of vertices where the special curve has multiplicity 0.

Suppose that the system of special adjoints of index k have the property that they are merely required to have a certain order, namely, $n-3k$, then the original curve can have no singular point of multiplicity greater than k. Let us find the necessary and sufficient conditions for the attainability of this:

THEOREM 1. *Given a curve whose special adjoint system of index k has the following properties:*

(1) The fundamental curves are rational;

(2) The base points of the adjoint system on the given curve comprise all the singularities of each fundamental curve and form a normal base, determining it uniquely;

(3) The total number of such base points is equal to the number of different fundamental curves;

(4) Through each base point on the given curve will pass at least two fundamental curves: then

The given curve can be transformed into one having no singularity of order higher than k.

Usually there are no base points for the adjoints other than the assigned singularities of the given curve, but such might conceivably arise. Now to prove the theorem. Suppose that we have a curve of order ν with singular points of orders $\rho_1\rho_2 \ldots$ which impose independent conditions on the curve. Then an immediate calculation shows that the expression

$$\nu(\nu + 3) - \Sigma\rho_i(\rho_i + 1)$$

is an invariant for the curve under all quadratic transformations, provided the summation be extended to include any new singular points introduced by the transformation. In the case of a rational curve uniquely determined by a normal base including all of its singularities we have

$$\begin{gathered}\nu(\nu + 3) - \Sigma\rho_i(\rho_i + 1) = 0\\ \nu(\nu - 3) - \Sigma\rho_i(\rho_i - 1) = -2\\ 3\nu - 1 - \Sigma\rho_i = 0\\ \nu(\nu - 3l) = \Sigma\rho_i(\rho_i - l) - (l - 1).\end{gathered}$$

This equation seems to tell us that the total number of intersections of the fundamental curve with a special adjoint of index l is less than the number at the singular points alone. Hence there can be no special adjoints of index greater than 1, nor of index 1 either, for the curve is rational by (1). Now it is known that if a curve be rational, the N.S. condition that it can be carried into a straight line by means of a Cremona transformation is that all special adjoint system become impossible.[2] The fundamental points of the quadratic transformations are singular points of this curve, and so by (2) of the curve of order n. Hence we may reduce one fundamental curve to be of order unity without introducing any new ones. This curve, or rather straight line, is determined by a normal base, two points. Through each will pass at least one other fundamental curve. One of these, if not a line, will have at least two other base points, for a curve or order above two which is rational cannot be determined by just two points. Hence we have a base point off of our line. Using this and the two base points on the line as fundamental in a quadratic transformation, we abolish the line as a fundamental curve of the system of index k, yet without introducing any new fundamental curve. Continuing thus we abolish all the fundamental curves, hence, by (3) we abolish all the base points. Then the system consists in the totality of curves of order $n-3k$, and the original curve had no singularity of order above k.

Suppose that, in particular, $k = 1$. Then we can transform our original curve to one with no singular points. Conversely, suppose we have a non-singular curve. If the special adjoint system of index 1 do not exist, it is rational and a line or conic, and none of the subsequent systems exist. If it be not rational, these conditions are fulfilled.

THEOREM 2. *The necessary and sufficient condition that it be possible to transform a curve by a Cremona transformation into one with no singular points is that either it is rational with no special adjoints of any index, or that the conditions of theorem 1 are fulfilled for the adjoint system of index 1.*

Take $k = 2$. Then we can transform to a curve with no singular points but double points. Conversely, suppose we have a curve whose only singular points are double points, distinct or infinitely near. Then if the adjoint system of index 2 exists, these conditions are satisfied. If the system of index two does not exist, but that of index 1 does, then the curve is of order five or less, for a system of curves of order 0 with no assigned points may be said to exist. If of order 5 it is not hyperelliptic. Again, if the fundamental curve of the system of index 1 be a line or a conic or nothing at all, and if the curve be not hyperelliptic of deficiency 3, the curve is of order 5 or less but not a quintic with just a triple point, and it is easy to see that it can be carried into a curve with only singular points of the second order.

THEOREM 3. *The necessary and sufficient condition that it be possible to transform a curve by a Cremona transformation into one with no singular points but double points, or with none at all, is that either it is rational lacking all special adjoint systems, or that it lacks the system of index 2 but that of index 1 can be transformed to a line or conic or nothing at all, yet the curve is not hyperelliptic of deficiency 3, or, lastly, the system of index 2 exists and obeys the conditions of theorem 1.*

It is well known that any curve can be carried on a Cremona transformation into one whose only singular points are ordinary ones, and by a birational transformation, that is to say, a transformation birational for the curve alone, into one whose only singular points are nodes. Our theorem 3 shows that, in general, this cannot be accomplished by a Cremona transformation. I do not know what additional restrictions beside those in 3 are necessary in order that the double points arrived at should all be distinct with distinct tangents.

[1] Conf. Enriques-Chesini: "Teoria geometrica delle equazioni," Vol. III, Bologna, **1924**, pp. 178 and 187.

[2] *Ibid.*, p. 188.